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Self-sustained solitary waves in non-equilibrium media

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The paper is devoted to the creation of an original model of propagation of weak but finite-amplitude waves initiating an exothermic process connected with chemical reaction or relaxation of a non-equilibrium medium. The medium is single phase (a fluid), or a homogeneous two-phase medium (liquid with gas bubbles). A nonlinear differential model describing wave–kinetic interaction and wave evolution is derived. The linear dispersion and dissipative features of these systems are investigated both analytically and numerically. Attention is paid to explanation of the physical mechanisms resulting in the formation of a self-sustained solitary wave, which in terms of synergetics could be called a 'dissipative structure'.

1. Introduction

A number of papers, starting from the classical work of Wood & Kirkwood (1957) have been devoted to the description of wave–kinetic interaction in non-equilibrium systems (with chemical reaction or relaxation). The phenomenon of acoustic disturbance amplification in media with spatio-homogeneous exothermal reaction or relaxation was analytically described for the first time by Nakoryakov & Borissov (1976) and Clarke (1977). The exact quasi-linear evolution wave equation for irreversibly reacting media was derived by Abouseif & Toong (1981), and an approximate equation for the evolution of high-frequency quasi-linear disturbances was integrated numerically. In the present paper, the authors develop the original model of Nakoryakov & Borissov (1976), examining the evolution of one-dimensional weak (but not linear) waves in a non-equilibrium medium and deriving the criteria for steady-state solitary wave formation.

2. Substantiation of the model

Let us analyse the processes in a system such as a chemically inert liquid containing bubbles of a chemically active gaseous mixture. (This problem also includes the case of a single-phase fluid medium.) A wave compresses and adiabatically heats the gas in the bubble, initiating an exothermic reaction in the gaseous phase. One can describe the wave as a planar one if the size of the bubbles is much less than the distance between them and that distance is much less than the length of the wave. The process in the gaseous phase is then governed by the following equations:
the equation of state (in general form)

$$\frac{\mathrm{d}\rho_g}{\mathrm{d}t}=\left[\left(\frac{\partial\rho_g}{\partial p_g}\right)_{S_g}\right]_f\frac{\mathrm{d}p_g}{\mathrm{d}t}+\left[\left(\frac{\partial\rho_g}{\partial S_g}\right)_{p_g}\right]_f\frac{\mathrm{d}S_g}{\mathrm{d}t}+\frac{\mathrm{d}Y}{\mathrm{d}t}\left(\frac{\partial\rho_g}{\partial Y}\right)_{S_g,p_g}; \tag{1}$$

the kinetic equation for the one-step chemical transformation reactants → products (in general form)

$$\omega(p_g, S_g, Y) = \rho_g(p_g, S_g, Y)\frac{\mathrm{d}Y}{\mathrm{d}t}; \tag{2}$$

the equation of entropy production (due to change of chemical potential only)

$$\frac{\mathrm{d}S_g}{\mathrm{d}t} = \frac{q}{T_g}\frac{\mathrm{d}Y}{\mathrm{d}t}. \tag{3}$$

In (1)–(3) ρ is the density, p is the pressure, T is the temperature, S is the entropy, Y is the mass fraction of the products, ω is the reaction rate, q is the specific heat release of the reaction, subscript g denotes the gaseous phase, and subscript f corresponds to the condition $Y = \text{Const}$ ('frozen reaction'). Transport processes in the gaseous phase are neglected. One cannot neglect the variation of gas entropy here, unlike the case of equilibrium systems.

Taking into account (2) and (3), equation (1) can be rewritten in the form

$$F = C_f^{-2}\frac{\mathrm{d}p_g}{\mathrm{d}t} - \frac{\mathrm{d}\rho_g}{\mathrm{d}t}, \tag{4}$$

where
$$F = \omega Q; \quad Q = \frac{q\rho_g \sigma_f}{c_{pf}} + \frac{\Delta v_g}{v_g}; \quad v_g = \rho_g^{-1};$$

$$C_f^2 = \left[\left(\frac{\partial p_g}{\partial \rho_g}\right)_{S_g}\right]_f; \quad c_{pf} = \left[\left(T_g\frac{\partial S_g}{\partial T_g}\right)_{p_g}\right]_f; \quad \sigma_f = \left(\frac{\partial v_g}{\partial T_g}\right)_f.$$

Here C_f is the speed of high-frequency disturbances in the gas, c_p is the specific heat capacity of the gas mixture at constant pressure, σ is the temperature expansion coefficient of the gas, v is the specific volume, and Δv_g is the difference in specific volumes of the gas mixture after and before the reaction.

Using (2) and (3), the identity

$$\frac{\mathrm{d}F}{\mathrm{d}t} = F_p\frac{\mathrm{d}p_g}{\mathrm{d}t} + F_S\frac{\mathrm{d}S_g}{\mathrm{d}t} + F_Y\frac{\mathrm{d}Y}{\mathrm{d}t}$$

can be rewritten
$$\frac{\mathrm{d}F}{\mathrm{d}t} = F_p\frac{\mathrm{d}p_g}{\mathrm{d}t} + \left(\frac{qF_S}{Q\rho_g T_g} + \frac{F_Y}{Q\rho_g}\right)F,$$

where
$$F_p = \left[\left(\frac{\partial F}{\partial p_g}\right)_{S_g}\right]_f; \quad F_S = \left[\left(\frac{\partial F}{\partial S_g}\right)_{p_g}\right]_f; \quad F_Y = \left(\frac{\partial F}{\partial Y}\right)_{p_g, S_g}.$$

Eliminating F from the last relation with the help of (4), the following equation of state can be derived for the gaseous phase:

$$\frac{\tau}{N}C_f^2\frac{\mathrm{d}}{\mathrm{d}t}\left[C_f^{-2}\frac{\mathrm{d}p_g}{\mathrm{d}t} - \frac{\mathrm{d}\rho_g}{\mathrm{d}t}\right] + \frac{\mathrm{d}p_g}{\mathrm{d}t} - C^2\frac{\mathrm{d}\rho_g}{\mathrm{d}t} = 0, \tag{5}$$

where
$$N = (1+L)(1-\delta); \quad L = 1 + \frac{qF_S}{T_g F_Y}; \quad \delta = \frac{\tau F_p C_f^2}{1+L}$$

$$\tau = -\frac{Q\rho_g}{F_Y}; \quad C^2 = \frac{C_f^2}{1-\delta};$$

$|\tau|$ and C play the roles of characteristic reaction time and low-frequency disturbance speed in the gas; L is a dimensionless parameter connected with irreversibility of the process, and δ is the dimensionless parameter of chemical dispersion. Generally τ, C, L, N, δ are variables determined by the kinetic equation (2). Equation (5) was derived for the first time in Nakoryakov & Borissov (1976). We assume below that the parameters of chemical dispersion and chemical irreversibility are small ($|\delta| \ll 1, |L| \ll 1$), and consequently $N \sim 1$.

Usually the partial derivations of F have definite signs: $F_S > 0, F_p > 0, F_Y < 0$. So for this usual kinetic dependence we obtain $C^2 > C_f^2$ for $q > 0$ (exothermic reaction) and $C^2 < C_f^2$ for $q < 0$.

If the compression of the gas by a weak wave in an adiabatic process, then the propagation speed of the wave can be approximately written as the first two terms of a series:

$$C_f^2 = \left(1+\frac{\gamma+1}{2}\frac{\rho_g-\rho_{g0}}{\rho_{g0}}\right)C_{f0}^2+o(\epsilon), \tag{6}$$

where
$$C_{f0}^2 = \left(\gamma\frac{p_g}{\rho_g}\right)_0; \quad \frac{\rho_g-\rho_{g0}}{\rho_{g0}} = O(\epsilon);$$

subscript 0 denotes the unperturbed state of the medium, γ is the adiabatic exponent (γ = Const), and ϵ is a small parameter ($0 < \epsilon \ll 1$).

Under the supposition of weak bubble pulsations in the wave, one can use the linearized equation of bubble dynamics, which describes the added mass effect of the liquid:

$$\frac{p_g-p}{\rho_l} = r_0\frac{\mathrm{d}^2r}{\mathrm{d}t^2},$$

where r is the bubble radius, subscript l denotes the liquid phase, and the variables without subscripts g or l correspond to the two-phase medium.

The density ρ of the two-phase medium is determined by the volume fraction ϕ or mass fraction χ of the gas (χ = Const):

$$\rho = \rho_l(1-\phi)+\rho_g\phi = \left(\frac{1-\chi}{\rho_l}+\frac{\chi}{\rho_g}\right)^{-1}; \quad \rho_l = \text{Const}.$$

It is easy then to find the expression for weak fluctuations of ρ_g:

$$\frac{\partial\rho}{\partial\rho_g} = \frac{\partial}{\partial\rho_g}\left(\frac{\rho_l}{1-\chi+\chi\rho_l/\rho_g}\right) = \chi\frac{\rho^2}{\rho_g^2} = \phi\frac{\rho}{\rho_g},$$

i.e.
$$(\rho_g-\rho_{g0})/\rho_{g0} = (\rho-\rho_0)/(\phi_0\rho_0)+o(\epsilon). \tag{7}$$

Neglecting the fluctuations of ϕ, one can also write the connection between pressure in the bubble and pressure in two-phase medium as

$$\frac{\mathrm{d}p_g}{\mathrm{d}t} = \frac{\mathrm{d}p}{\mathrm{d}t}-\beta\frac{\mathrm{d}^3\rho}{\mathrm{d}t^3}, \tag{8}$$

where $\beta = r_0/[3\phi_0(1-\phi_0)]$ is the bubble dispersion parameter.

Combining (5)–(8), one can now derive an equation governing the evolution of a weak wave in a two-phase reactive medium:

$$\frac{\tau}{N}a_{f0}^2\left(1+\frac{\gamma+1}{2\phi_0}\frac{\rho-\rho_0}{\rho_0}\right)\frac{\mathrm{d}}{\mathrm{d}t}\left[a_{f0}^{-2}\left(1+\frac{\gamma+1}{2\phi_0}\frac{\rho-\rho_0}{\rho_0}\right)^{-1}\left(\frac{\mathrm{d}p}{\mathrm{d}t}-\beta\frac{\mathrm{d}^3\rho}{\mathrm{d}t^3}\right)-\frac{\mathrm{d}\rho}{\mathrm{d}t}\right]$$
$$+\frac{\mathrm{d}p}{\mathrm{d}t}-\beta\frac{\mathrm{d}^3\rho}{\mathrm{d}t^3}-a^2\left(1+\frac{\gamma+1}{2\phi_0}\frac{\rho-\rho_0}{\rho_0}\right)\frac{\mathrm{d}\rho}{\mathrm{d}t}+o(\epsilon^2) = 0, \tag{9}$$

where
$$\frac{a_{f0}^2}{C_{f0}^2} = \frac{a^2 C_f^2}{C^2 C_{f0}^2} = \frac{\rho_{g0}}{\phi_0 \rho_0};$$

a_f and a correspond to the velocities of high- and low-frequency (relatively to the inverse reaction time) disturbances in the bubbly medium. (All the disturbances are supposed to be of low frequency relative to the frequency of free bubble pulsations. To reduce (9) to the single-phase case one need only to formally set $\beta = 0$, $\phi_0 = 1$.)

To obtain a closed system of equations one has to introduce the laws of mass and momentum conservation for two-phase medium,

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho \frac{\partial u}{\partial x}, \tag{10}$$

$$\rho \frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{\partial p}{\partial x} + \eta \frac{\partial^2 u}{\partial x^2}, \tag{11}$$

where u is the mass velocity of the medium and η is the viscosity of the liquid. The viscosity of the liquid is important, being one of the main dissipative mechanisms.

To transform (9)–(11) to dimensionless form it is convenient to use the following reference quantities:
$$\rho^* = \rho_0; \quad u^* = a_{f0}; \quad t^* = \tau.$$

Note that even for high-amplitude waves the density of a two-phase medium with low gas fraction varies only insignificantly, and the variations of mass velocity are correspondingly small. So it is quite natural to assume the following:

$$\frac{\rho - \rho_0}{\rho^*} = \hat{\rho} = O(\epsilon), \quad \frac{u}{u^*} = \hat{u} = O(\epsilon), \quad \frac{p}{\rho^* u^{*2}} = \hat{p} = O(\epsilon)$$

($\rho^* u^{*2} \gg p_0$), and additionally to suppose the dispersion of the medium to be weak:

$$\frac{\eta}{\rho^* u^{*2} t^*} = \hat{\eta} = O(\epsilon), \quad \frac{\beta}{(u^* t^*)^2} = \hat{\beta} = O(\epsilon).$$

For the case of a single-phase fluid medium, one needs the parameter variations to be much less than their values in the unperturbed state. Below, dimensionless variables are used everywhere.

The system of dimensionless equations (9)–(11) is the basis for derivation of a single equation describing the structure and dynamics of a wave travelling in one direction. We will neglect all terms $O(\epsilon^s)$, $s > 2$.

Weak dispersion of the medium allows one to use the method of slowly changing wave profile in a translating coordinate frame:

$$y = \hat{x} - \hat{t}, \quad \xi = \epsilon \hat{t} \quad (\xi \text{ is the `slow time'});$$

$$\frac{\partial}{\partial \hat{x}} = \frac{\partial}{\partial y}; \quad \frac{\partial}{\partial \hat{t}} = \epsilon \frac{\partial}{\partial \xi} - \frac{\partial}{\partial y} = \frac{\partial}{\partial z} - \frac{\partial}{\partial y}; \quad \left(\frac{\partial}{\partial z} = O(\epsilon)\right);$$

giving

$$\frac{1}{N}\left[\hat{p}_{yy} - 2\hat{p}_{zy} - 2\hat{u}\hat{p}_{yy} - \hat{\beta}\hat{\rho}_{yyyy} + 2\hat{\rho}_{zy} - \hat{\rho}_{yy} + 2\hat{u}\hat{\rho}_{yy} - \frac{\gamma+1}{2\phi_0}(\hat{p}_y\hat{\rho}_y + \hat{\rho}\hat{\rho}_{yy})\right]$$
$$+ \hat{p}_z - \hat{p}_y + \hat{u}\hat{p}_y + \hat{\beta}\hat{\rho}_{yyy} + \alpha\left(\hat{\rho}_y - \hat{\rho}_z - \hat{u}\hat{\rho}_y + \frac{\gamma+1}{2\phi_0}\hat{\rho}\hat{\rho}_y\right) + o(\epsilon^2) = 0, \tag{12}$$

$$\hat{\rho}_z - \hat{\rho}_y + \hat{u}\hat{\rho}_y + \hat{u}_y + \hat{\rho}\hat{u}_y + o(\epsilon^2) = 0, \tag{13}$$

$$\hat{u}_y - \hat{p}_y - \hat{u}_z + \hat{\eta}\hat{u}_{yy} + \hat{\rho}\hat{u}_y - \hat{u}\hat{u}_y + o(\epsilon^2) = 0. \tag{14}$$

Substituting for $\hat{\rho}$ and $\hat{u}$ in terms of $\hat{p}$ with the help of (13) and (14), we can reduce (12) to the following form:

$$\left(1-\frac{1}{N}\frac{\partial}{\partial y}\right)[2\hat{p}_z-\hat{\eta}\hat{p}_{yy}+\hat{\beta}\hat{p}_{yyy}+\bar{\Psi}(\hat{p}^2)_y]+(\alpha-1)\hat{p}_y+o(\epsilon^2)=0, \tag{15}$$

where $\alpha = a^2/a_f^2$ and $\bar{\Psi} = 1+(\gamma+1)/2\phi_0$ is the coefficient of nonlinearity; the first term of $\bar{\Psi}$ corresponds to the usual fluid dynamic steepening nonlinearity and the last one described 'bubble' nonlinearity. Equation (15) is derived using the conditions of weak chemical dispersion and irreversibility:

$$|\alpha-1| = \left|\frac{1+L}{N}-1\right| \approx |\delta| = O(\epsilon).$$

We emphasize that (15) is derived under the condition of small-amplitude disturbances of p, u, ρ, ϕ, but this equation does not include any assumption on the nonlinearity of chemical kinetic law. So one can conclude that (15) supplemented by (2) must describe exactly the evolution of small (but finite)-amplitude waves initiating, in the medium, an irreversible reaction through an arbitrary nonlinear kinetic law. The medium can be a gas–liquid bubbly medium or a pure fluid medium.

3. Dynamics of linear disturbances

Let us analyse the solution of the linearized version of (15). Eliminating fluid dynamic, bubble and chemical nonlinearities ($\bar{\Psi} = 0$, $\tau =$ Const, $\alpha =$ Const, $N =$ Const), we seek a solution in the form of the superposition of non-interacting harmonics:

$$\hat{p}_j = \sum_{j=0}^{\infty} P_j \exp(\Omega_j z + \mathrm{i}k_j y).$$

So for a single harmonic we have the dispersion relation

$$\Omega = \frac{\left(\dfrac{\alpha-1}{N}-\hat{\eta}\right)k^2-\dfrac{\hat{\eta}}{N^2}k^4}{2\left(1+\dfrac{k^2}{N^2}\right)}+\mathrm{i}\frac{\hat{\beta}k^3\left(1+\dfrac{k^2}{N^2}\right)-(\alpha-1)k}{2\left(1+\dfrac{k^2}{N^2}\right)}. \tag{16}$$

High-frequency limiting case ($k^2 \sim O(\epsilon^{-1})$) (reaction is 'frozen'):

$$\Omega \rightarrow \mathrm{i}\tfrac{1}{2}\hat{\beta}k^3-\tfrac{1}{2}\hat{\eta}k^2. \tag{17}$$

Dispersion relation (17) corresponds to the linear limit of the well-known Korteweg–de Vries–Burgers equation, which describes the propagation of waves in a chemically inert dissipative bubbly medium. According to (17) linear disturbances travel through the medium with the 'frozen' sound speed a_f. The profile of the solution can be monotonic or oscillating, depending on the relation between the parameters of the equation (Nakoryakov, Pokusaev & Shreiber 1983).

Low-frequency limiting case ($k^2 \sim O(\epsilon)$). Now the wave–kinetic interaction is significant (chemical dispersion terms are included in the relation $\Omega(k)$):

$$\Omega = \left(\frac{\alpha-1}{2N}-\frac{\hat{\eta}}{2}\right)k^2+\mathrm{i}\left(\frac{\hat{\beta}}{2}k^3-\frac{\alpha-1}{2}k\right). \tag{18}$$

Relation (18) also corresponds to the linearized Korteweg–de Vries–Burgers equation, but the coefficient $((\alpha-1)/2N-\frac{1}{2}\hat{\eta})$ before the second derivative can be either negative

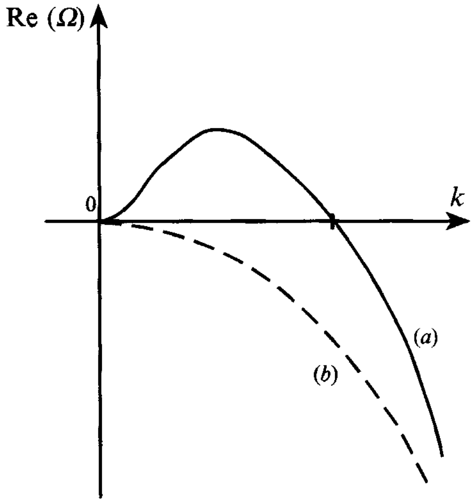

FIGURE 1. The curves $\mathrm{Re}(\Omega(k))$: (a) $(\alpha-1)/N > \hat{\eta}$; ($b$) $(\alpha-1)/N < \hat{\eta}$.

or positive. The latter implies the possibility of low-frequency harmonic amplification within the zone of chemical transformation. The amplification takes place as the effect of reaction thermal release predominates over that of energy dissipation (see figure 1):

$$\frac{\alpha-1}{N} > \hat{\eta}. \tag{19}$$

Note that (18) describes the propagation of linear disturbances with velocity a. In front of and behind the chemical transformation zone $a = a_f$, and (18) can be reduced to (17). But at the reaction zone a can exceed the value of the 'frozen' sound speed, owing to irreversibility of the system (as was shown before). In the case

$$\alpha > 1 \tag{20}$$

the velocity of low-frequency harmonics at the reaction zone is higher than the velocity of any harmonic behind the reaction zone. As a result it is possible for the chemical transformation zone to separate from the original wave, forming a solitary autowave. Note that the condition of separation, (20), is more stringent than the criterion of amplification, (19), because for exothermic reactions ($q > 0$) the parameter $\alpha - 1/N$ is always positive; its sign does not change with the sign of $(\alpha - 1)$, at least as long as

$$\left[\left(\frac{\partial F}{\partial p_g}\right)_{S_g}\right]_f > 0.$$

The regime with amplification but without autowave separation has been studied theoretically, and observed experimentally earlier. The criteria (19) and (20) together determine the possibility of formation of a self-sustained solitary wave: they show the connection of the physical mechanism leading to formation of the autowave with the chemical dispersion of the system.

4. Nonlinearity

The amplification of low-frequency harmonics and formation of a solitary autowave are linear effects appearing owing to chemical dispersion. In turn, according to the evolution equation (15), nonlinear interaction between harmonics could feed energy

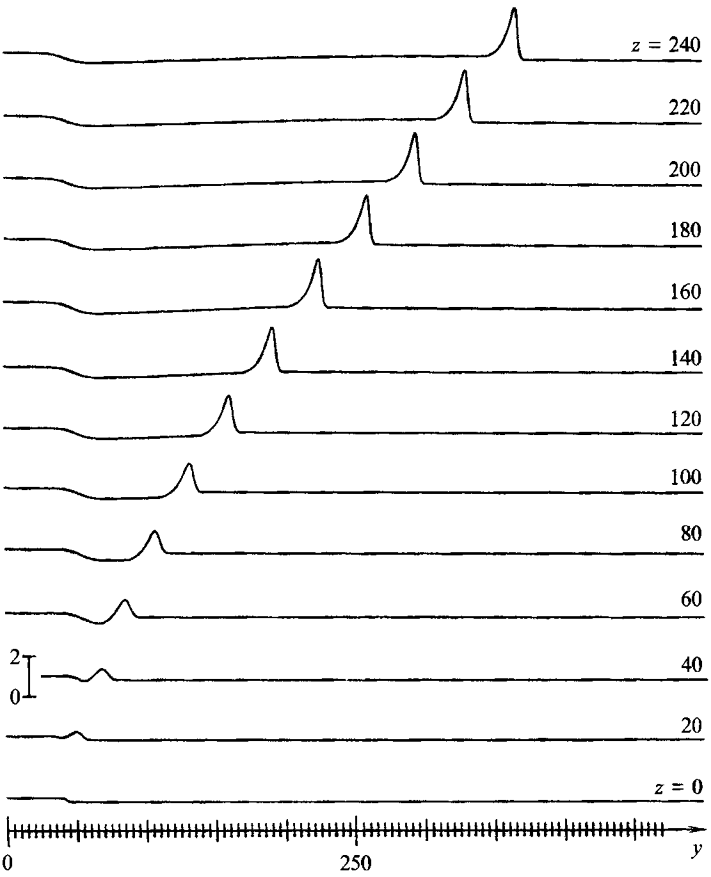

FIGURE 2. The evolution of a wave with $W_i = 0.2$.

from unstable low-frequency disturbances to attenuated high-frequency ones. Ultimately, a stationary solitary wave spectrum would be possible. A similar mechanism of stabilization of the solution is known for a number of problems, related to the self-organization theory: as examples we quote the front structure of a laminar flame; the surface of a liquid film flowing down an inclined wall; the solid surface sublimated by a laser; the solidification front structure of a weak alloy; multi-headed gaseous detonations, and others. Appearing in these systems are so-called 'dissipative structures' which exist in the steady-state region owing to energy exchange with the medium, and the decay of high-frequency harmonics is caused by various physical mechanisms: thermal conductivity, surface tension, viscosity, real-gas-dynamic effects, etc.

In spite of the relative simplicity of the nonlinear equation (15), we have failed to find its exact analytical solutions (even in the case $N = \text{Const}$, $\alpha = \text{Const}$). Analytical solutions are known only for essentially more simple equations with Burgers–Korteweg–de Vries nonlinearity: the Burgers equation, Korteweg–de Vries equation, Korteweg–de Vries–Burgers equation, Kuramoto–Sivashinsky equation (Kudriashov 1990; Minaev 1992). So the further investigation of solutions of (15) could only be carried out using numerical modelling.

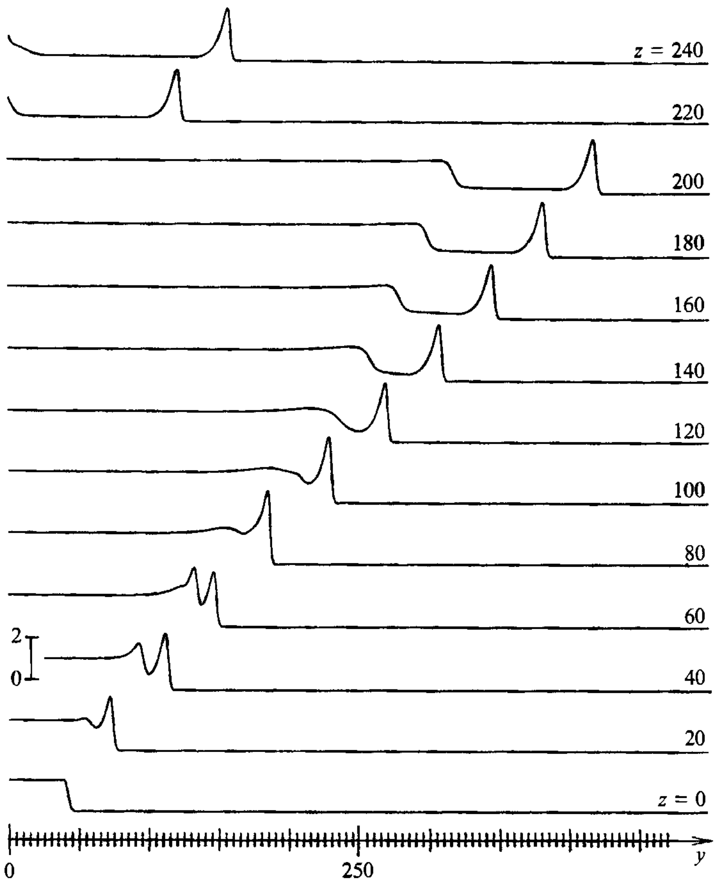

FIGURE 3. The evolution of a wave with $W_\iota = 1.5$.

5. Results of numerical modelling

The aim of the illustrative numerical modelling carried out here was to verify whether the conclusions on the possibility of solitary wave formation made on the basis of linear theory are true or not. Owing to the linearity of this process, the system of equations (15), (2) was simplified assuming the model of linear chemical dispersion. So the values of variables τ, N and α were set to be constants. Bubble dispersion and bubble nonlinearity were not taken into account, i.e. the medium was supposed to be a single-phase fluid. Thus the model (15), (2) retained only the mechanisms of linear chemical dispersion, viscous dissipation and the usual fluid dynamic steepening nonlinearity (the last is necessary for the existence of a steady-state solution). We took

$$\tau = \text{Const}, \quad N = \text{Const}, \quad \alpha = \surd 3, \quad \hat{\beta} = 0, \quad \hat{\eta} = 2.6, \quad \bar{\Psi} = 1.$$

These values of the coefficients were chosen to satisfy the criteria (19) and (20) in order to illustrate the mechanism of solitary autowave formation. (In practical calculations the chemical and bubble nonlinear properties of the medium could of course be significant, and should not be neglected, unlike this illustrative numerical solution.)

Equation (15) was approximated by finite differences, with approximation error about $\tilde{\tau}+\tilde{h}^2$ ($\tilde{\tau} = \frac{1}{30}$ is the time step, $\tilde{h} = \frac{1}{3}$ is the distance between mesh points). The stability of the calculation algorithm was analysed carefully. The precision of the

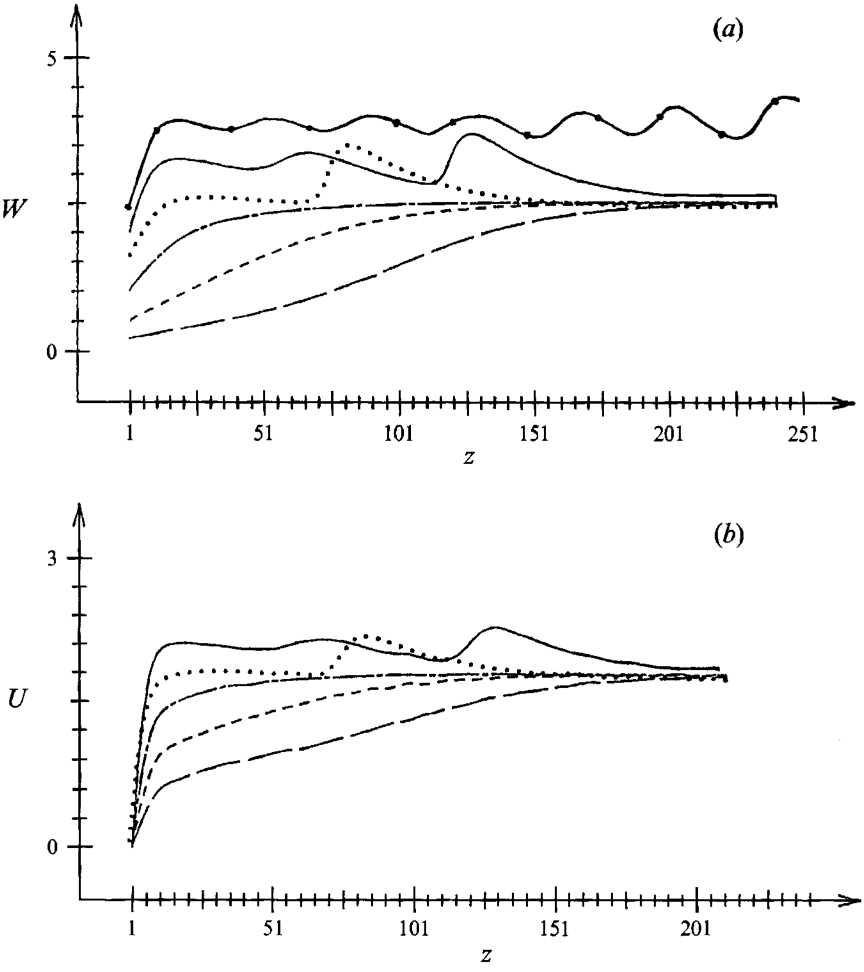

FIGURE 4. The evolution of (*a*) amplitude W and (*b*) velocity U of the leading shock, for W_i equal to: 0.2 (———), 0.5 (–––––), 1.0 (—·—·—), 1.5 (·········), 2.0 (——), 2.5 (————).

calculations was checked by using the analytical solution of Burgers equation (as $\alpha = 1$). Owing to the assumption $\tau = \text{Const}$ the size of the reaction zone was determined by τ multiplied by the calculated velocity of the wave front. So if we defined the point where the reaction switches on as the point where the local amplitude of the leading wave was equal to the amplitude of the original wave (W_i), then we could also calculate the point where the reaction switches off. Outside the reaction zone (where $\alpha = 1$) the usual Burgers equation was solved; the whole equation was solved only inside the reaction zone.

A step-like wave was chosen as an initial condition; its amplitude was changed from $W_i = 0.2$ to $W_i = 2.5$. The waves with $W_i \leqslant 2.0$ developed in different ways, but resulted in the formation of a solitary wave, whose parameters did not depend on W_i: its amplitude was approximately equal to 2.5, and velocity (in the translating coordinate frame) was approximately equal to 1.8, see figures 2–4. As W_i increased, the stationary autowave formed more quickly. If W_i was equal to or exceeded the value of the solitary autowave amplitude, then the separation of a self-sustained structure did not occur; see figures 4 and 5.

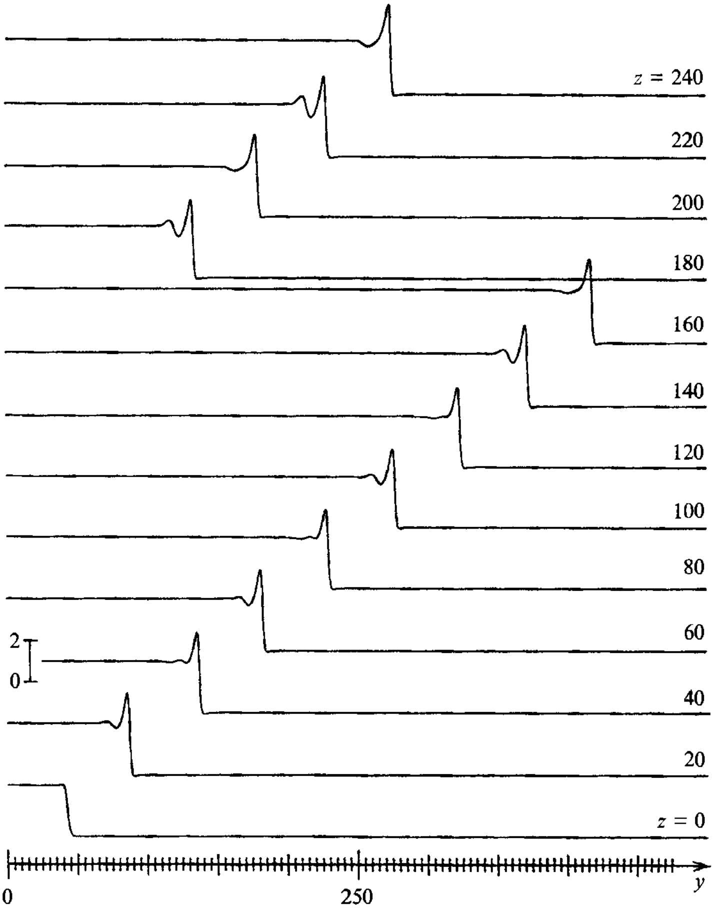

FIGURE 5. The evolution of a wave with $W_i = 2.5$.

6. Discussion and conclusions

The model (15), (2) derived in this paper describes the effects of linear dispersion and nonlinearity of chemical kinetics, of bubble and fluid dynamics on small- but finite-amplitude wave propagation.

The linear analytical and numerical results explain the physical mechanism for formation of a solitary autowave in a system where a wave initiates a non-equilibrium process. These autowaves appear under certain conditions, described by criteria (19) and (20); the following are necessary:

(i) at the reaction zone the amplification of low-frequency disturbances takes place;

(ii) the low-frequency (relative to the inverse reaction time) linear disturbance velocity at the reaction zone exceeds the 'frozen' sound speed in the medium;

(iii) the original wave amplitude is less than that of the stationary autowave.

By the mechanism of propagation this autowave is related to detonation waves. In practice a weak shock wave could effectively initiate the reaction if the medium were first heated by a strong shock wave. Unfortunately, there are no data on such experiments, so the authors could not carry out a quantitative comparison of theoretical and experimental results. Qualitatively the comparison could be carried out using data on gas–liquid detonation (Sychev 1985; Gülhan 1989). Unlike our model, the condition of weak variation of bubble radius does not hold in these experiments.

Owing to nonlinear pulsations large peak pressures and temperatures are achieved in the bubbles, which significantly influences the value of the characteristic reaction time. So the initiation of reaction is determined not only by the shock wave but also by the inertia of the added mass of the liquid. In spite of this distinction the model derived here qualitatively describes the formation of a solitary self-sustained wave (gas–liquid detonation) wave in detail. This surprising fact allows us to conclude that the physical mechanism of solitary autowave appearance is not determined by the bubble dispersion (unlike the classical solitons), and is connected mainly with the chemical dispersion, dissipative and nonlinear features of the system. This mechanism is the same both for bubbly and single-phase media. But in a gas–liquid medium the formation of a stationary solitary autowave can be observed experimentally much more easily owing to the low value of the sound speed in hetero-phase systems.

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